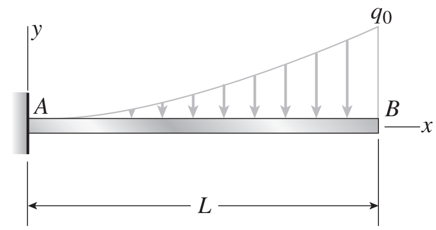
Problem 9.5-11 Determine the angle of rotation θ_B and deflection δ_B at the free end of a cantilever beam AB supporting a parabolic load defined by the equation $q = q_0x^2/L^2$ (see figure).



Solution 9.5-11 **Cantilever beam (parabolic load)**

LOAD: $q = \frac{q_0 x^2}{L^2}$ qdx = element of load

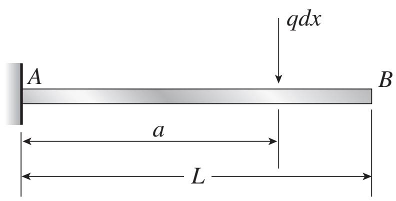


TABLE G-1, CASE 5 (Set a equal to x)

$$\theta_B = \int_0^L \frac{(qdx)(x^2)}{2EI} = \frac{1}{2EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right)x^2dx$$

$$= \frac{q_0}{2EIL^2}\int_0^L x^4dx = \frac{q_0L^3}{10EI} \quad \longleftarrow$$

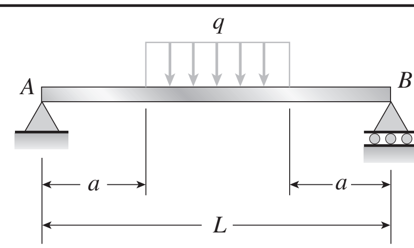
$$\delta_B = \int_0^L \frac{(qdx)(x^2)}{6EI}(3L - x)$$

$$= \frac{1}{6EI}\int_0^L \left(\frac{q_0x^2}{L^2}\right)(x^2)(3L - x)\,dx$$

$$= \frac{q_0}{6EIL^2}\int_0^L (x^4)(3L - x)\,dx = \frac{13q_0L^4}{180EI} \quad \longleftarrow$$

Problem 9.5-12 A simple beam AB supports a uniform load of intensity q acting over the middle region of the span (see figure).

Determine the angle of rotation θ_A at the left-hand support and the deflection δ_{max} at the midpoint.



Solution 9.5-12 **Simple beam (partial uniform load)**

LOAD: qdx = element of load

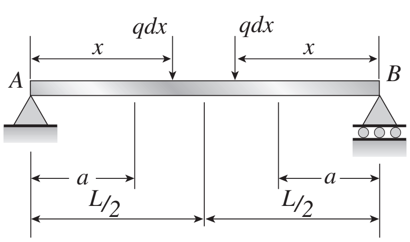


TABLE G-2, CASE 6 $\theta_A = \frac{Pa(L - a)}{2EI}$

Replace P by qdx

Replace a by x

Integrate x from a to $L/2$

$$\theta_A = \int_a^{L/2} \frac{qdx}{2EI}(x)(L - x) = \frac{q}{2EI}\int_a^{L/2}(xL - x^2)\,dx$$

$$= \frac{q}{24EI}(L^3 - 6a^2L + 4a^3) \quad \longleftarrow$$

TABLE G-2, CASE 6 $\delta_{max} = \frac{Pa}{24EI}(3L^2 - 4a^2)$

Replace P by qdx

Replace a by x

Integrate x from a to $L/2$

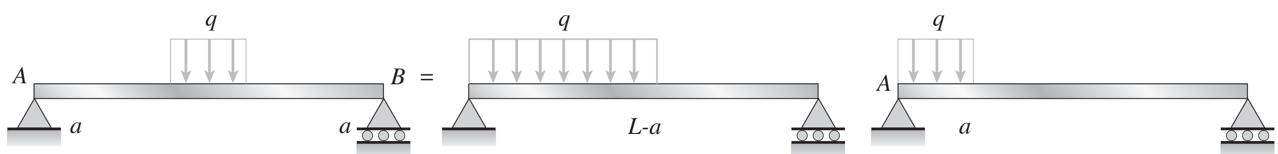
$$\delta_{max} = \int_a^{L/2} \frac{qdx}{24EI}(x)(3L^2 - 4x^2)$$
$$= \frac{q}{24EI}\int_a^{L/2}(3L^2x - 4x^3)\,dx$$
$$= \frac{q}{384EI}(5L^4 - 24a^2L^2 + 16a^4) \longleftarrow$$

ALTERNATE SOLUTION (not recommended; algebra is extremely lengthy)

Table G-2, Case 3

$$\theta_A = \frac{q(L-a)^2}{24LEI}[2L - (L-a)]^2 - \frac{qa^2}{24LEI}(2L-a)^2$$
$$= \frac{q}{24EI}(L^3 - 6La^2 + 4a^3) \longleftarrow$$

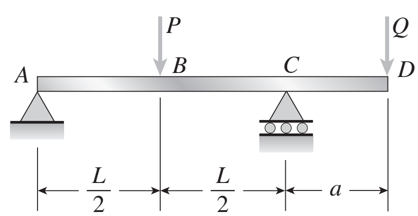
$$\delta_{max} = \frac{q(L/2)}{24LEI}\Big[(L-a)^4 - 4L(L-a)^3 + 4L^2(L-a)^2 + 2(L-a)^2\left(\frac{L}{2}\right)^2 - 4L(L-a)\left(\frac{L}{2}\right)^2 + L\left(\frac{L}{2}\right)^3\Big]$$
$$= \frac{qa^2}{24LEI}\Big[-La^2 + 4L^2\left(\frac{L}{2}\right) + a^2\left(\frac{L}{2}\right) - 6L\left(\frac{L}{2}\right)^2 + 2\left(\frac{L}{2}\right)^3\Big]$$
$$\delta_{max} = \frac{q}{384EI}(5L^4 - 24L^3a^2 + 16a^4) \longleftarrow$$



Problem 9.5-13 The overhanging beam $ABCD$ supports two concentrated loads P and Q (see figure).

(a) For what ratio P/Q will the deflection at point B be zero?

(b) For what ratio will the deflection at point D be zero?



Solution 9.5-13 Overhanging beam

(a) DEFLECTION AT POINT B

Table G-2, Cases 4 and 7

$$\delta_B = \frac{PL^3}{48EI} - Qa\left(\frac{L^2}{16EI}\right) = 0 \qquad \frac{P}{Q} = \frac{3a}{L} \longleftarrow$$

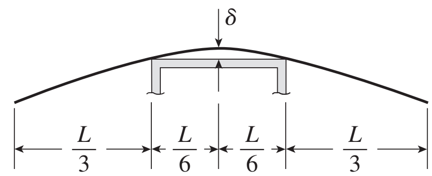
(b) DEFLECTION AT POINT D

Table G-2, Case 4; Table G-1, Case 4; Table G-2, Case 7

$$\delta_D = -\frac{PL^2}{16EI}(a) + \frac{Qa^3}{3EI} + Qa\left(\frac{L}{3EI}\right)(a) = 0$$
$$\frac{P}{Q} = \frac{16a(L+a)}{3L^2} \longleftarrow$$

Problem 9.5-14 A thin metal strip of total weight W and length L is placed across the top of a flat table of width $L/3$ as shown in the figure.

What is the clearance δ between the strip and the middle of the table? (The strip of metal has flexural rigidity EI.)



Solution 9.5-14 Thin metal strip

W = total weight $\quad q = \dfrac{W}{L}$

EI = flexural rigidity

FREE BODY DIAGRAM (the part of the strip above the table)

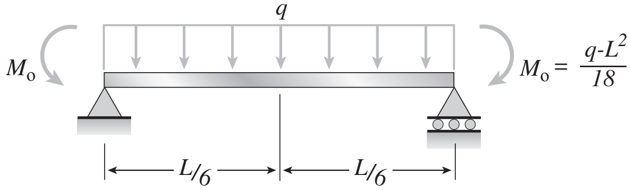


TABLE G-2, CASES 1 AND 10

$$\delta = -\frac{5q}{384EI}\left(\frac{L}{3}\right)^4 + \frac{M_0}{8EI}\left(\frac{L}{3}\right)^2$$

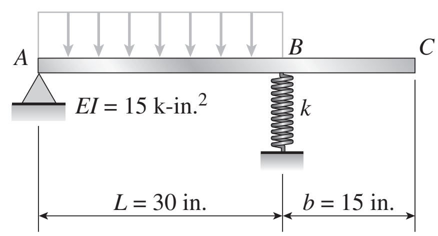
$$= -\frac{5qL^4}{31{,}104EI} + \frac{qL^4}{1296EI}$$

$$= \frac{19qL^4}{31{,}104EI}$$

But $q = \dfrac{W}{L}$: $\quad \therefore \delta = \dfrac{19WL^3}{31{,}104EI}$ ←

Problem 9.5-15 An overhanging beam ABC with flexural rigidity EI = 15 k-in.^{2} is supported by a pin support at A and by a spring of stiffness k at point B (see figure). Span AB has length L = 30 in. and carries a uniformly distributed load. The overhang BC has length b = 15 in.

For what stiffness k of the spring will the uniform load produce no deflection at the free end C?



Solution 9.5-15 Overhanging beam with a spring support

EI = 15 k-in.^{2} $\quad L$ = 30 in. $\quad b$ = 15 in.
q = intensity of uniform load

(1) Assume that point B is on a simple support

Table G-2, Case 1

$$\delta'_C = \theta_B b = \frac{qL^3}{24EI}(b) \quad \text{(upward deflection)}$$

(2) Assume that the spring shortens

$$R_B = \text{force in the spring} = \frac{qL}{2}$$

$$\delta_B = \frac{R_B}{k} = \frac{qL}{2k}$$

$$\delta''_C = \delta_B\left(\frac{L+b}{L}\right) = \frac{q}{2k}(L+b) \quad \text{(downward deflection)}$$

(3) Deflection at point C (equal to zero)

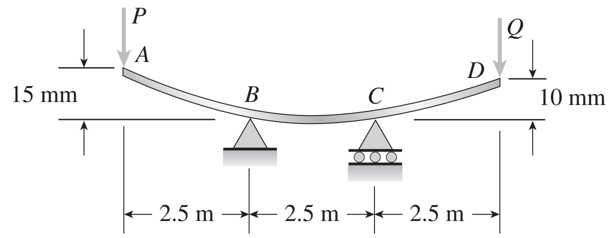
$$\delta_C = \delta'_C - \delta''_C = \frac{qL^3 b}{24EI} - \frac{q}{2k}(L+b) = 0$$

Solve for k: $\quad k = \dfrac{12EI}{L^3}\left(1 + \dfrac{L}{b}\right)$ ←

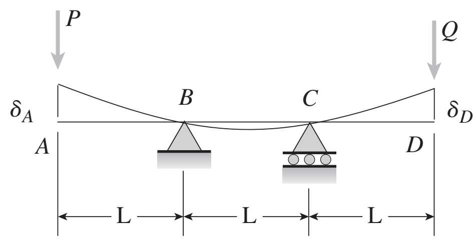
Substitute numerical values: k = 20 lb/in. ←

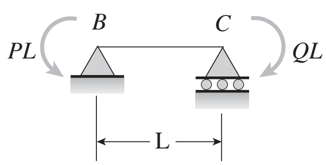
Problem 9.5-16 A beam $ABCD$ rests on simple supports at B and C (see figure). The beam has a slight initial curvature so that end A is 15 mm above the elevation of the supports and end D is 10 mm above.

What loads P and Q, acting at points A and D, respectively, will move points A and D downward to the level of the supports? (The flexural rigidity EI of the beam is 2.5×10^6 N · m^2.)



Solution 9.5-16 **Beam with initial curvature**





$\delta_A = 15$ mm
$\delta_D = 10$ mm
$EI = 2.5 \times 10^6$ N · m^2
$L = 2.5$ m

Table G-2, Case 7: $\theta_B = PL\left(\frac{L}{3EI}\right) + QL\left(\frac{L}{6EI}\right)$

$$= \frac{L^2}{6EI}(2P + Q)$$

Table G-1, Case 4: $\delta_A = \frac{PL^3}{3EI} + \theta_B L = \frac{L^3}{6EI}(4P + Q)$

$$4P + Q = \frac{6EI\delta_A}{L^3} \qquad \text{(Eq. 1)}$$

In a similar manner, $\delta_D = \frac{L^3}{6EI}(4Q + P)$

$$4Q + P = \frac{6EI\delta_D}{L^3} \qquad \text{(Eq. 2)}$$

Solve Eqs. (1) and (2):

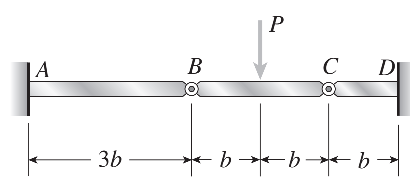
$$P = \frac{2EI}{5L^3}(4\delta_A - \delta_D) \quad Q = \frac{2EI}{5L^3}(4\delta_D - \delta_A) \quad \leftarrow$$

Substitute numerical values:

$P = 3200$ N $\quad Q = 1600$ N $\leftarrow$

Problem 9.5-17 The compound beam $ABCD$ shown in the figure has fixed supports at ends A and D and consists of three members joined by pin connections at B and C.

Find the deflection δ under the load P.



Solution 9.5-17 **Compound beam**

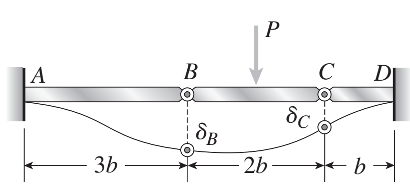


Table G-1, Case 4 and Table G-2, Case 4

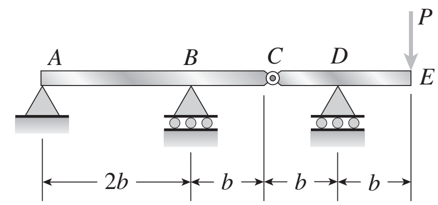
$$\delta_B = \frac{"PL^3"}{3EI} = \left(\frac{P}{2}\right)(3b)^3\left(\frac{1}{3EI}\right) = \frac{9Pb^3}{2EI}$$

$$\delta_C = \frac{"PL^3"}{3EI} = \left(\frac{P}{2}\right)(b^3)\left(\frac{1}{3EI}\right) = \frac{Pb^3}{6EI}$$

$$\delta = \frac{1}{2}(\delta_B + \delta_C) + \frac{P(2b)^3}{48EI} = \frac{5Pb^3}{2EI} \quad \leftarrow$$

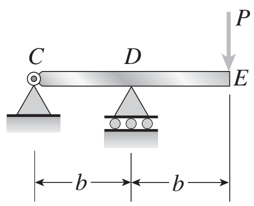
Problem 9.5-18 A compound beam *ABCDE* (see figure) consists of two parts (*ABC* and *CDE*) connected by a hinge at *C*.

Determine the deflection δ_E at the free end *E* due to the load *P* acting at that point.



Solution 9.5-18 Compound beam

BEAM *CDE* WITH A SUPPORT AT *C*

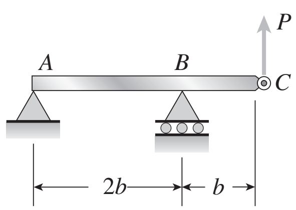


δ'_E = downward deflection of point *E*

$$\delta'_E = \frac{Pb^3}{3EI} + \theta'_D\, b = \frac{Pb^3}{3EI} + Pb\left(\frac{b}{3EI}\right)b$$

$$= \frac{2Pb^3}{3EI}$$

BEAM *ABC*



δ_C = upward deflection of point *C*

$$\delta_C = \frac{Pb^3}{3EI} + Q_B b = \frac{Pb^3}{3EI} + Pb\left(\frac{2b}{3EI}\right)b$$

$$= \frac{Pb^3}{EI}$$

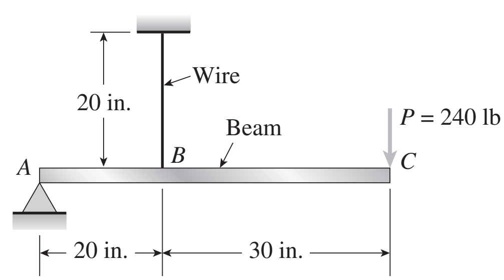
The upward deflection δ_C produces an equal downward displacement at point *E*. $\therefore\ \delta''_E = \delta_C = \dfrac{Pb^3}{EI}$

DEFLECTION AT END *E*

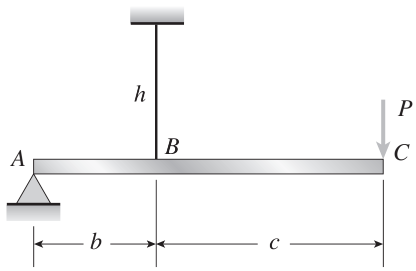
$$\delta_E = \delta'_E + \delta''_E = \frac{5Pb^3}{3EI} \quad \longleftarrow$$

Problem 9.5-19 A steel beam *ABC* is simply supported at *A* and held by a high-strength steel wire at *B* (see figure). A load $P = 240$ lb acts at the free end *C*. The wire has axial rigidity $EA = 1500 \times 10^3$ lb, and the beam has flexural rigidity $EI = 36 \times 10^6$ lb-in.^{2}

What is the deflection δ_C of point *C* due to the load *P*?



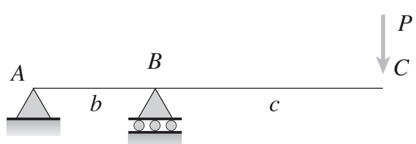
Solution 9.5-19 Beam supported by a wire



$P = 240$ lb $b = 20$ in. $c = 30$ in. $h = 20$ in.

Beam: $EI = 36 \times 10^6$ lb-in.^{2}
Wire: $EA = 1500 \times 10^3$ lb

(1) ASSUME THAT POINT B IS ON A SIMPLE SUPPORT



$$\delta'_C = \frac{Pc^3}{3EI} + \theta'_B c = \frac{Pc^3}{3EI} + (Pc)\left(\frac{b}{3EI}\right)c$$

$$= \frac{Pc^2}{3EI}(b + c) \quad \text{(downward)}$$

(2) ASSUME THAT THE WIRE STRETCHES

T = tensile force in the wire

$$= \frac{P}{b}(b + c)$$

$$\delta_B = \frac{Th}{EA} = \frac{Ph(b + c)}{EAb}$$

$$\delta''_C = \delta_B\left(\frac{b + c}{b}\right) = \frac{Ph(b + c)^2}{EAb^2} \quad \text{(downward)}$$

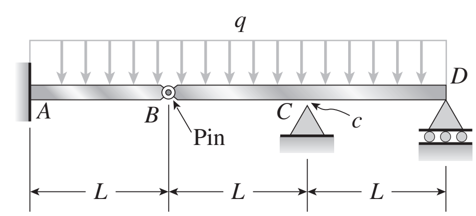
(3) DEFLECTION AT POINT C

$$\delta_C = \delta'_C + \delta''_C \quad = P(b + c)\left[\frac{c^2}{3EI} + \frac{h(b + c)}{EAb^2}\right] \quad \longleftarrow$$

Substitute numerical values:
$\delta_C = 0.10$ in. $+ 0.02$ in. $= 0.12$ in. $\longleftarrow$

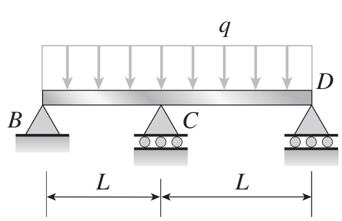
Problem 9.5-20 The compound beam shown in the figure consists of a cantilever beam AB (length L) that is pin-connected to a simple beam BD (length $2L$). After the beam is constructed, a clearance c exists between the beam and a support at C, midway between points B and D. Subsequently, a uniform load is placed along the entire length of the beam.

What intensity q of the load is needed to close the gap at C and bring the beam into contact with the support?



Solution 9.5-20 Compound beam

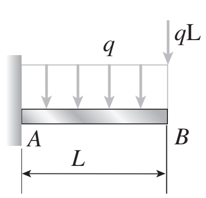
BEAM BCD WITH A SUPPORT AT B



$$\delta'_C = \frac{5q(2L)^4}{384EI}$$

$$= \frac{5qL^4}{24EI}$$

CANTILEVER BEAM AB



$$\delta_B = \frac{qL^4}{8EI} + \frac{(qL)L^3}{3EI}$$

$$= \frac{11qL^4}{24EI} \quad \text{(downward)}$$

δ''_C = downward displacement of point C due to δ_B

$$\delta''_C = \frac{1}{2}\,\delta_B = \frac{11qL^4}{48EI}$$

DOWNWARD DISPLACEMENT OF POINT C

$$\delta_C = \delta'_C + \delta''_C = \frac{5qL^4}{24EI} + \frac{11qL^4}{48EI} = \frac{7qL^4}{16EI}$$

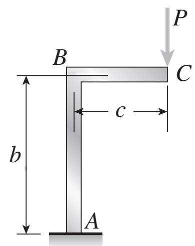
c = clearance $c = \delta_C = \dfrac{7qL^4}{16EI}$

INTENSITY OF LOAD TO CLOSE THE GAP

$$q = \frac{16EIc}{7L^4} \quad \longleftarrow$$

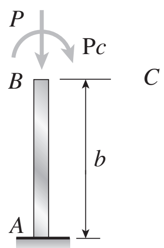
Problem 9.5-21 Find the horizontal deflection δ_h and vertical deflection δ_v at the free end C of the frame ABC shown in the figure. (The flexural rigidity EI is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the load P.



Solution 9.5-21 **Frame** *ABC*

MEMBER *AB*:



δ_h = horizontal deflection of point B

Table G-1, Case 6:

$$\delta_h = \frac{(Pc)b^2}{2EI} = \frac{Pcb^2}{2EI}$$

$$\theta_B = \frac{Pcb}{EI}$$

Since member BC does not change in length, δ_h is also the horizontal displacement of point C.

$$\therefore \ \delta_h = \frac{Pcb^2}{2EI} \quad \longleftarrow$$

MEMBER *BC* WITH *B* FIXED AGAINST ROTATION

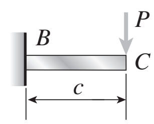


Table G-1, Case 4:

$$\delta'_C = \frac{Pc^3}{3EI}$$

VERTICAL DEFLECTION OF POINT *C*

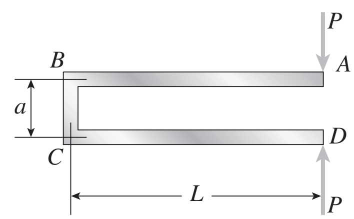
$$\delta_C = \delta_v = \delta'_C + \theta_B c = \frac{Pc^3}{3EI} + \frac{Pcb}{EI}(c)$$

$$= \frac{Pc^2}{3EI}(c + 3b)$$

$$\delta_v = \frac{Pc^2}{3EI}(c + 3b) \quad \longleftarrow$$

Problem 9.5-22 The frame $ABCD$ shown in the figure is squeezed by two collinear forces P acting at points A and D. What is the decrease δ in the distance between points A and D when the loads P are applied? (The flexural rigidity EI is constant throughout the frame.)

Note: Disregard the effects of axial deformations and consider only the effects of bending due to the loads P.



Solution 9.5-22 **Frame** *ABCD*

MEMBER *BC*:

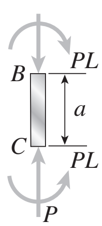


Table G-2, Case 10: $\theta_B = \frac{(PL)a}{2EI} = \frac{PLa}{2EI}$

MEMBER *BA*:

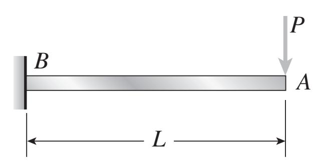


Table G-1, Case 4: $\delta_A = \frac{PL^3}{3EI} + \theta_B L$

$$= \frac{PL^3}{3EI} + \frac{PLa}{2EI}(L)$$

$$= \frac{PL^2}{6EI}(2L + 3a)$$

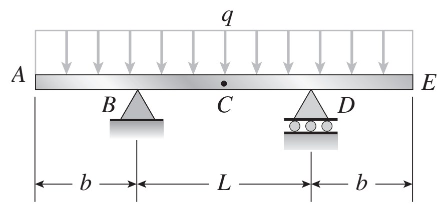
DECREASE IN DISTANCE BETWEEN POINTS *A* AND *D*

$$\delta = 2\delta_A = \frac{PL^2}{3EI}(2L + 3a) \quad \longleftarrow$$

Problem 9.5-23 A beam *ABCDE* has simple supports at *B* and *D* and symmetrical overhangs at each end (see figure). The center span has length *L* and each overhang has length *b*. *A* uniform load of intensity *q* acts on the beam.

(a) Determine the ratio *b*/*L* so that the deflection δ_C at the midpoint of the beam is equal to the deflections δ_A and δ_E at the ends.

(b) For this value of *b*/*L*, what is the deflection δ_C at the midpoint?



Solution 9.5-23 Beam with overhangs

BEAM *BCD*:

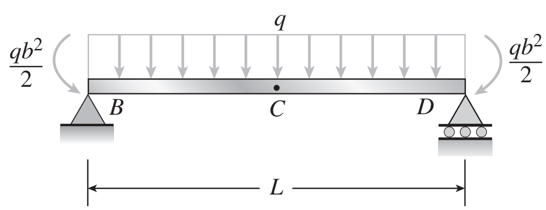


Table G-2, Case 1 and Case 10:

$$\theta_B = \frac{qL^3}{24EI} - \frac{qb^2}{2}\left(\frac{L}{2EI}\right) = \frac{qL}{24EI}(L^2 - 6b^2)$$

(clockwise is positive)

$$\delta_C = \frac{5qL^4}{384EI} - \frac{qb^2}{2}\left(\frac{L^2}{8EI}\right) = \frac{qL^2}{384EI}(5L^2 - 24b^2) \quad (1)$$

(downward is positive)

BEAM *AB:*

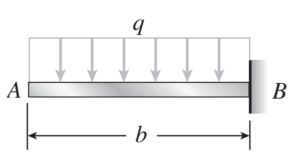


Table G-1, Case 1:

$$\delta_A = \frac{qb^4}{8EI} - \theta_B b = \frac{qb^4}{8EI} - \frac{qL}{24EI}(L^2 - 6b^2)b$$

$$= \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

(downward is positive)

DEFLECTION δ_C EQUALS DEFLECTION δ_A

$$\frac{qL^2}{384EI}(5L^2 - 24b^2) = \frac{qb}{24EI}(3b^3 + 6b^2L - L^3)$$

Rearrange and simplify the equation:

$$48b^4 + 96b^3L + 24b^2L^2 - 16bL^3 - 5L^4 = 0$$

or

$$48\left(\frac{b}{L}\right)^4 + 96\left(\frac{b}{L}\right)^3 + 24\left(\frac{b}{L}\right)^2 - 16\left(\frac{b}{L}\right) - 5 = 0$$

(a) RATIO $\frac{b}{L}$

Solve the preceding equation numerically:

$$\frac{b}{L} = 0.40301 \qquad \text{Say, } \frac{b}{L} = 0.4030 \quad \longleftarrow$$

(b) DEFLECTION δ_C (EQ. 1)

$$\delta_C = \frac{qL^2}{384EI}(5L^2 - 24b^2)$$

$$= \frac{qL^2}{384EI}[5L^2 - 24\,(0.40301\,L)^2]$$

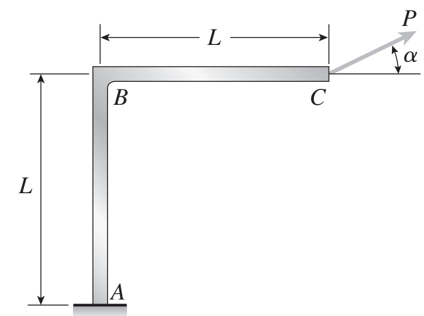
$$= 0.002870\,\frac{qL^4}{EI} \quad \longleftarrow$$

(downward deflection)

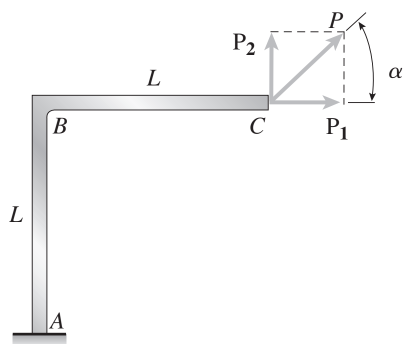
Problem 9.5-24 A frame ABC is loaded at point C by a force P acting at an angle α to the horizontal (see figure). Both members of the frame have the same length and the same flexural rigidity.

Determine the angle α so that the deflection of point C is in the same direction as the load. (Disregard the effects of axial deformations and consider only the effects of bending due to the load P.)

Note: A direction of loading such that the resulting deflection is in the same direction as the load is called a *principal direction*. For a given load on a planar structure, there are two principal directions, perpendicular to each other.



Solution 9.5-24 Principal directions for a frame



P_1 and P_2 are the components of the load P

$P_1 = P\cos\alpha$

$P_2 = P\sin\alpha$

IF P_1 ACTS ALONE $\quad \delta'_H = \dfrac{P_1L^3}{3EI}$ (to the right)

$$\delta'_v = \theta_B L = \left(\frac{P_1L^2}{2EI}\right)L = \frac{P_1L^3}{2EI}$$

(downward)

IF P_2 ACTS ALONE $\quad \delta''_H = \dfrac{P_2L^3}{2EI}$ (to the left)

$$\delta''_v = \frac{P_2L^3}{3EI} + \theta_B L = \frac{P_2L^3}{3EI} + \left(\frac{P_2L^2}{EI}\right)L = \frac{4P_2L^3}{3EI}$$

(upward)

DEFLECTIONS DUE TO THE LOAD P

$$\delta_H = \frac{P_1L^3}{3EI} - \frac{P_2L^3}{2EI} = \frac{L^3}{6EI}(2P_1 - 3P_2) \text{ (to the right)}$$

$$\delta_v = -\frac{P_1L^3}{2EI} + \frac{4P_2L^3}{3EI} = \frac{L^3}{6EI}(-3P_1 + 8P_2) \text{ (upward)}$$

$$\frac{\delta_v}{\delta_H} = \frac{-3P_1 + 8P_2}{2P_1 - 3P_2} = \frac{-3P\cos\alpha + 8P\sin\alpha}{2P\cos\alpha - 3P\sin\alpha} = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

PRINCIPAL DIRECTIONS

The deflection of point C is in the same direction as the load P.

$$\therefore \tan\alpha = \frac{P_2}{P_1} = \frac{\delta_v}{\delta_H} \quad \text{or} \quad \tan\alpha = \frac{-3 + 8\tan\alpha}{2 - 3\tan\alpha}$$

Rearrange and simplity: $\tan^2\alpha + 2\tan\alpha - 1 = 0$ (quadratic equation)

Solving, $\tan\alpha = -1 \pm \sqrt{2}$

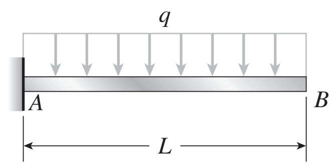
$\alpha = 22.5°, \quad 112.5°, \quad -67.5°, \quad -157.5°,$ ←

Moment-Area Method

The problems for Section 9.6 are to be solved by the moment-area method. All beams have constant flexural rigidity EI.

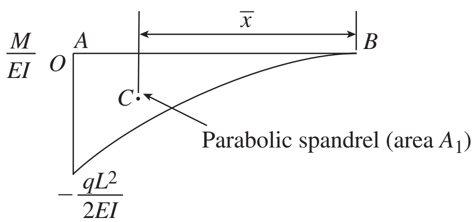
Problem 9.6-1 A cantilever beam AB is subjected to a uniform load of intensity q acting throughout its length (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.



Solution 9.6-1 Cantilever beam (uniform load)

M/EI DIAGRAM:



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 18: $A_1 = \frac{1}{3}(L)\left(\frac{qL^2}{2EI}\right) = \frac{qL^3}{6EI}$

$$\bar{x} = \frac{3L}{4}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{qL^3}{6EI}$$

$$\theta_A = 0 \qquad \theta_B = \frac{qL^3}{6EI} \quad \text{(clockwise)}$$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

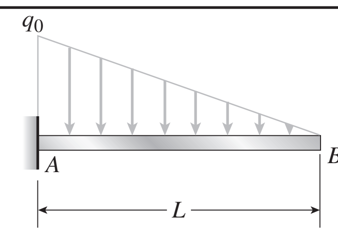
$$Q_1 = A_1\bar{x} = \left(\frac{qL^3}{6EI}\right)\left(\frac{3L}{4}\right) = \frac{qL^4}{8EI}$$

$$\delta_B = Q_1 = \frac{qL^4}{8EI} \quad \text{(Downward)}$$

(These results agree with Case 1, Table G-1.)

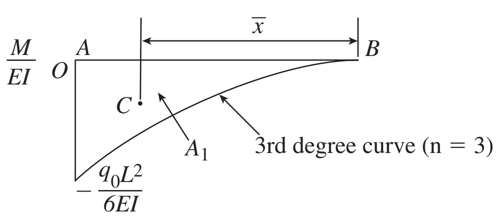
Problem 9.6-2 The load on a cantilever beam AB has a triangular distribution with maximum intensity q_0 (see figure).

Determine the angle of rotation θ_B and the deflection δ_B at the free end.



Solution 9.6-2 Cantilever beam (triangular load)

M/EI DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas.

Appendix D, Case 20:

$$A_1 = \frac{bh}{n+1} = \frac{1}{4}(L)\left(\frac{q_0L^2}{6EI}\right) = \frac{q_0L^3}{24EI}$$

$$\bar{x} = \frac{b(n+1)}{n+2} = \frac{4L}{5}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = \frac{q_0L^3}{24\,EI}$$

$$\theta_A = 0 \qquad \theta_B = \frac{q_0L^3}{24EI} \quad \text{(clockwise)}$$

DEFLECTION

Q_1 = First moment of area A_1 with respect to B

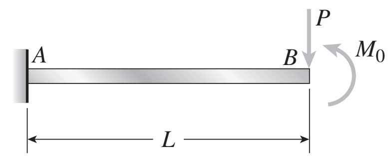
$$Q_1 = A_1\bar{x} = \left(\frac{q_0L^3}{24EI}\right)\left(\frac{4L}{5}\right) = \frac{q_0L^4}{30EI}$$

$$\delta_B = Q_1 = \frac{q_0L^4}{30EI} \quad \text{(Downward)}$$

(These results agree with Case 8, Table G-1.)

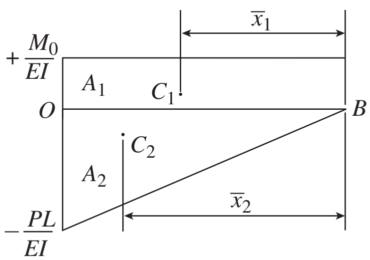
Problem 9.6-3 A cantilever beam AB is subjected to a concentrated load P and a couple M_0 acting at the free end (see figure).

Obtain formulas for the angle of rotation θ_B and the deflection δ_B at end B.



Solution 9.6-3 Cantilever beam (force P and couple M_0)

M/EI DIAGRAM



NOTE: A_1 is the *M/EI* diagram for M_0 (rectangle). A_2 is the *M/EI* diagram for P (triangle).

ANGLE OF ROTATION

Use the sign conventions for the moment-area theorems (page 628 of textbook).

$$A_1 = \frac{M_0 L}{EI} \quad \bar{x}_1 = \frac{L}{2} \quad A_2 = -\frac{PL^2}{2EI} \quad \bar{x}_2 = \frac{2L}{3}$$

$$A_0 = A_1 + A_2 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0 \qquad \theta_A = 0$$

$$\theta_B = A_0 = \frac{M_0 L}{EI} - \frac{PL^2}{2EI}$$

(θ_B is positive when counterclockwise)

DEFLECTION

Q = first moment of areas A_1 and A_2 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

$$t_{B/A} = Q = \delta_B \qquad \delta_B = \frac{M_0 L^2}{2EI} - \frac{PL^3}{3EI}$$

(δ_B is positive when upward)

FINAL RESULTS

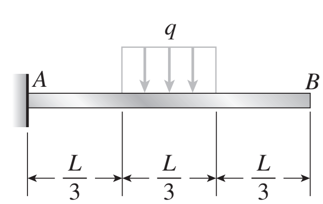
To match the sign conventions for θ_B and δ_B used in Appendix G, change the signs as follows.

$$\theta_B = \frac{PL^2}{2EI} - \frac{M_0 L}{EI} \text{ (positive clockwise)} \longleftarrow$$

$$\delta_B = \frac{PL^3}{3EI} - \frac{M_0 L^2}{2EI} \text{ (positive downward)} \longleftarrow$$

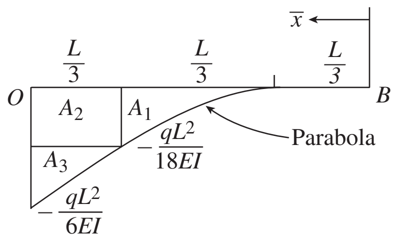
(These results agree with Cases 4 and 6, Table G-1.)

Problem 9.6-4 Determine the angle of rotation θ_B and the deflection δ_B at the free end of a cantilever beam AB with a uniform load of intensity q acting over the middle third of the length (see figure).



Solution 9.6-4 Cantilever beam with partial uniform load

M/EI DIAGRAM



ANGLE OF ROTATION

Use absolute values of areas. Appendix D, Cases 1, 6, and 18:

$$A_1 = \frac{1}{3}\left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{162EI} \qquad \bar{x}_1 = \frac{L}{3} + \frac{3}{4}\left(\frac{L}{3}\right) = \frac{7L}{12}$$

$$A_2 = \left(\frac{L}{3}\right)\left(\frac{qL^2}{18EI}\right) = \frac{qL^3}{54EI} \qquad \bar{x}_2 = \frac{2L}{3} + \frac{L}{6} = \frac{5L}{6}$$

$$A_3 = \frac{1}{2}\left(\frac{L}{3}\right)\left(\frac{qL^2}{9EI}\right) = \frac{qL^3}{54EI} \qquad \bar{x}_3 = \frac{2L}{3} + \frac{2}{3}\left(\frac{L}{3}\right) = \frac{8L}{9}$$

$$A_0 = A_1 + A_2 + A_3 = \frac{7qL^3}{162EI}$$

$$\theta_{B/A} = \theta_B - \theta_A = A_0$$

$$\theta_A = 0 \qquad \theta_B = \frac{7qL^3}{162EI} \quad \text{(clockwise)} \quad \longleftarrow$$

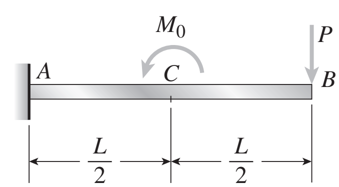
DEFLECTION

Q = first moment of area A_0 with respect to point B

$$Q = A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3 = \frac{23qL^4}{648EI}$$

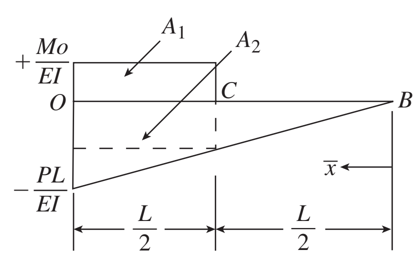
$$\delta_B = Q = \frac{23qL^4}{648EI} \quad \text{(Downward)} \quad \longleftarrow$$

Problem 9.6-5 Calculate the deflections δ_B and δ_C at points B and C, respectively, of the cantilever beam ACB shown in the figure. Assume M_0 = 36 k-in., P = 3.8 k, L = 8 ft, and $EI = 2.25 \times 10^9$ lb-in.^{2}



Solution 9.6-5 Cantilever beam (force P and couple M_0)

M/EI DIAGRAM



NOTE: A_1 is the *M/EI* diagram for M_0 (rectangle). A_2 is the *M/EI* diagram for P (triangle).

Use the sign conventions for the moment-area theorems (page 628 of textbook).

DEFLECTION δ_B

Q_B = first moment of areas A_1 and A_2 with respect to point B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{3L}{4}\right) - \frac{1}{2}\left(\frac{PL}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{L^2}{24EI}(9M_0 - 8PL)$$

$$t_{B/A} = Q_B = \delta_B \qquad \delta_B = \frac{L^2}{24EI}(9M_0 - 8PL)$$

(δ_B is positive when upward)

DEFLECTION δ_C

Q_C = first moment of area A_1 and left-hand part of A_2 with respect to point C

$$= \left(\frac{M_0}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) - \frac{1}{2}\left(\frac{PL}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{L^2}{48EI}(6M_0 - 5PL)$$

$$t_{C/A} = Q_C = \delta_C \qquad \delta_C = \frac{L^2}{48EI}(6M_0 - 5PL)$$

(δ_C is positive when upward)

ASSUME DOWNWARD DEFLECTIONS ARE POSITIVE
(change the signs of δ_B and δ_C)

$$\delta_B = \frac{L^2}{24EI}(8PL - 9M_0) \quad \longleftarrow$$

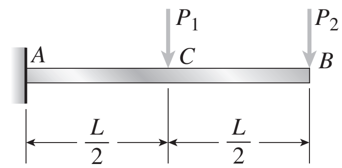
$$\delta_C = \frac{L^2}{48EI}(5PL - 6M_0) \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:
$M_0 = 36$ k-in. $P = 3.8$ k
$L = 8$ ft $= 96$ in. $EI = 2.25 \times 10^6$ k-in.^{2}

$\delta_B = 0.4981$ in. $- 0.0553$ in. $= 0.443$ in. $\longleftarrow$
$\delta_C = 0.1556$ in. $- 0.0184$ in. $= 0.137$ in. $\longleftarrow$

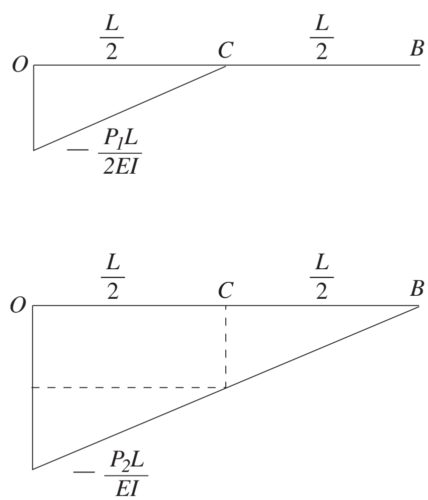
Problem 9.6-6 A cantilever beam ACB supports two concentrated loads P_1 and P_2 as shown in the figure.

Calculate the deflections δ_B and δ_C at points B and C, respectively. Assume $P_1 = 10$ kN, $P_2 = 5$ kN, $L = 2.6$ m, $E = 200$ GPa, and $I = 20.1 \times 10^6$ mm^4.



Solution 9.6-6 **Cantilever beam (forces P_1 and P_2)**

M/*EI* DIAGRAMS



$P_1 = 10$ kN $P_2 = 5$ kN $L = 2.6$ m
$E = 200$ GPa $I = 20.1 \times 10^6$ mm^4

Use absolute values of areas.

DEFLECTION δ_B

$\delta_B = t_{B/A} = Q_B =$ first moment of areas with respect to point B

$$\delta_B = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{2} + \frac{L}{3}\right) + \frac{1}{2}\left(\frac{P_2L}{EI}\right)(L)\left(\frac{2L}{3}\right)$$

$$= \frac{5P_1L^3}{48EI} + \frac{P_2L^3}{3EI} \quad \text{(downward)} \quad \longleftarrow$$

DEFLECTION δ_C

$\delta_C = t_{C/A} = Q_C =$ first moment of areas to the left of point C with respect to point C

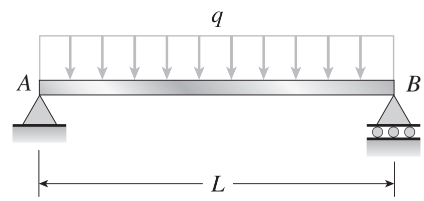
$$\delta_c = \frac{1}{2}\left(\frac{P_1L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right) + \left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{4}\right) + \frac{1}{2}\left(\frac{P_2L}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{3}\right)$$

$$= \frac{P_1L^3}{24EI} + \frac{5P_2L^3}{48EI} \quad \text{(downward)} \quad \longleftarrow$$

SUBSTITUTE NUMERICAL VALUES:

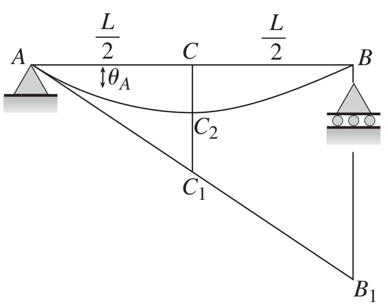
$\delta_B = 4.554$ mm $+ 7.287$ mm $= 11.84$ mm $\longleftarrow$
$\delta_C = 1.822$ mm $+ 2.277$ mm $= 4.10$ mm $\longleftarrow$
(deflections are downward)

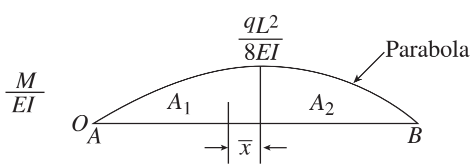
Problem 9.6-7 Obtain formulas for the angle of rotation θ_A at support A and the deflection δ_{max} at the midpoint for a simple beam AB with a uniform load of intensity q (see figure).



Solution 9.6-7 Simple beam with a uniform load

DEFLECTION CURVE AND *M/EI* DIAGRAM





δ_{max} = maximum deflection (distance CC_2)

Use absolute values of areas.

ANGLE OF ROTATION AT END A

Appendix D, Case 17:

$$A_1 = A_2 = \frac{2}{3}\left(\frac{L}{2}\right)\left(\frac{qL^2}{8EI}\right) = \frac{qL^3}{24EI}$$

$$\bar{x}_1 = \frac{3}{8}\left(\frac{L}{2}\right) = \frac{3L}{16}$$

$t_{B/A} = BB_1 =$ first moment of areas A_1 and A_2 with respect to point B

$$= (A_1 + A_2)\left(\frac{L}{2}\right) = \frac{qL^4}{24EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{qL^3}{24EI} \text{ (clockwise)} \longleftarrow$$

DEFLECTION δ_{max} AT THE MIDPOINT C

$$\text{Distance } CC_1 = \frac{1}{2}(BB_1) = \frac{qL^4}{48EI}$$

$t_{C_2}/A = C_2C_1 =$ first moment of area A_1 with respect to point C

$$= A_1\bar{x}_1 = \left(\frac{qL^3}{24EI}\right)\left(\frac{3L}{16}\right) = \frac{qL^4}{128EI}$$

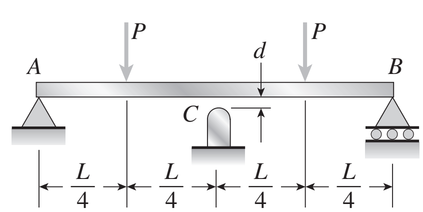
$$\delta_{max} = CC_2 = CC_1 - C_2C_1 = \frac{qL^4}{48EI} - \frac{qL^4}{128EI}$$

$$= \frac{5qL^4}{384EI} \text{ (downward)} \longleftarrow$$

(These results agree with Case 1 of Table G-2.)

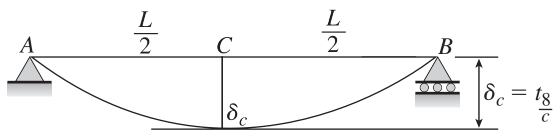
Problem 9.6-8 A simple beam AB supports two concentrated loads P at the positions shown in the figure. A support C at the midpoint of the beam is positioned at distance d below the beam before the loads are applied.

Assuming that $d = 10$ mm, $L = 6$ m, $E = 200$ GPa, and $I = 198 \times 10^6$ mm^4, calculate the magnitude of the loads P so that the beam just touches the support at C.

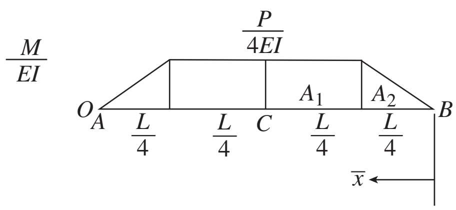


Solution 9.6-8 Simple beam with two equal loads

DEFLECTION CURVE AND M/EI DIAGRAM



δ_c = deflection at the midpoint C



$$A_1 = \frac{PL^2}{16EI} \quad \bar{x}_1 = \frac{3L}{8}$$

$$A_2 = \frac{PL^2}{32EI} \quad \bar{x}_2 = \frac{L}{6}$$

Use absolute values of areas.

DEFLECTION δ_c AT MIDPOINT OF BEAM

At point C, the deflection curve is horizontal.

$\delta_c = t_{B/C}$ = first moment of area between B and C with respect to B

$$= A_1\bar{x}_1 + A_2\bar{x}_2 = \frac{PL^2}{16EI}\left(\frac{3L}{8}\right) + \frac{PL^2}{32EI}\left(\frac{L}{6}\right)$$

$$= \frac{11PL^3}{384EI}$$

d = gap between the beam and the support at C

MAGNITUDE OF LOAD TO CLOSE THE GAP

$$\delta = d = \frac{11PL^3}{384EI} \qquad P = \frac{384EId}{11L^3} \quad \longleftarrow$$

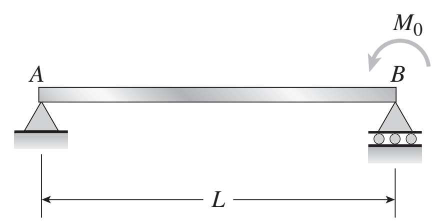
SUBSTITUTE NUMERICAL VALUES:

$d = 10$ mm $\quad L = 6$ m $\quad E = 200$ GPa

$I = 198 \times 10^6$ mm^4 $\quad P = 64$ kN $\longleftarrow$

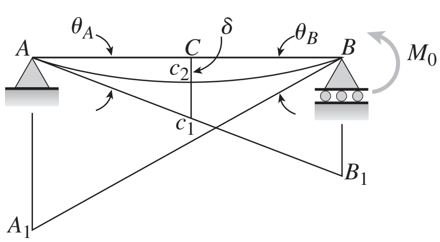
Problem 9.6-9 A simple beam AB is subjected to a load in the form of a couple M_0 acting at end B (see figure).

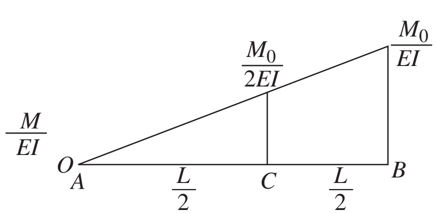
Determine the angles of rotation θ_A and θ_B at the supports and the deflection δ at the midpoint.



Solution 9.6-9 Simple beam with a couple M_0

DEFLECTION CURVE AND M/EI DIAGRAM





δ = deflection at the midpoint C

δ = distance CC_2

Use absolute values of areas.

ANGLE OF ROTATION θ_A

$t_{B/A} = BB_1$ = first moment of area between A and B with respect to B

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{L}{3}\right) = \frac{M_0L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0L}{6EI} \text{ (clockwise)} \quad \longleftarrow$$

ANGLE OF ROTATION θ_B

$t_{A/B} = AA_1 =$ first moment of area between A and B with respect to A

$$= \frac{1}{2}\left(\frac{M_0}{EI}\right)(L)\left(\frac{2L}{3}\right) = \frac{M_0 L^2}{3\,EI}$$

$$\theta_B = \frac{AA_1}{L} = \frac{M_0 L}{3EI} \text{ (Counterclockwise)} \longleftarrow$$

DEFLECTION δ AT THE MIDPOINT C

Distance $\quad CC_1 = \frac{1}{2}(BB_1) = \frac{M_0 L^2}{12EI}$

$t_{c_2}/A = C_2C_1 =$ first moment of area between A and C with respect to C

$$= \frac{1}{2}\left(\frac{M_0}{2EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = \frac{M_0 L^2}{48\,EI}$$

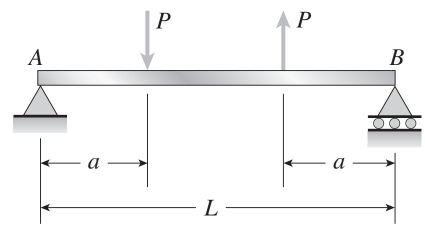
$$\delta = CC_1 - C_2C_1 = \frac{M_0 L^2}{12EI} - \frac{M_0 L^2}{48\,EI}$$

$$= \frac{M_0 L^2}{16EI} \quad \text{(Downward)} \longleftarrow$$

(These results agree with Case 7 of Table G-2.)

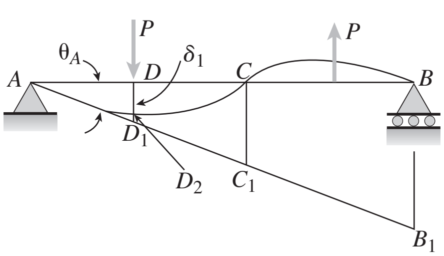
Problem 9.6-10 The simple beam AB shown in the figure supports two equal concentrated loads P, one acting downward and the other upward.

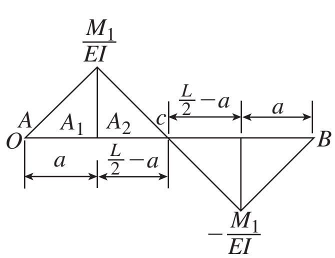
Determine the angle of rotation θ_A at the left-hand end, the deflection δ_1 under the downward load, and the deflection δ_2 at the midpoint of the beam.



Solution 9.6-10 Simple beam with two loads

Because the beam is symmetric and the load is antisymmetric, the deflection at the midpoint is zero.





$$\therefore \delta_2 = 0 \longleftarrow$$

$$\frac{M_1}{EI} = \frac{Pa(L-2a)}{LEI}$$

$$A_1 = \frac{1}{2}\left(\frac{M_1}{EI}\right)(a) = \frac{Pa^2(L-2a)}{2LEI}$$

$$A_2 = \frac{1}{2}\left(\frac{M_1}{EI}\right)\left(\frac{L}{2} - a\right) = \frac{Pa(L-2a)^2}{4LEI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{C/A} = CC_1 =$ first moment of area between A and C with respect to C

$$= A_1\left(\frac{L}{2} - a + \frac{a}{3}\right) + A_2\left(\frac{2}{3}\right)\left(\frac{L}{2} - a\right)$$

$$= \frac{Pa(L-a)(L-2a)}{12EI}$$

$$\theta_A = \frac{CC_1}{L/2} = \frac{Pa(L-a)(L-2a)}{6LEI} \text{ (clockwise)} \longleftarrow$$

DEFLECTION δ_1 UNDER THE DOWNWARD LOAD

Distance $DD_1 = \left(\frac{a}{L/2}\right)(CC_1)$

$$= \frac{Pa^2\,(L-a)\,(L-2a)}{6LEI}$$

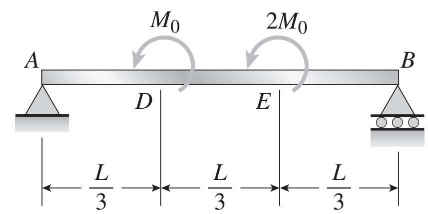
$t_{D_2}/A = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{a}{3}\right) = \frac{Pa^3(L-2a)}{6LEI}$$

$$\delta_1 = DD_1 - D_2D_1$$

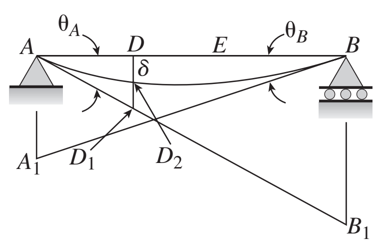
$$= \frac{Pa^2(L-2a)^2}{6LEI} \quad \text{(Downward)} \longleftarrow$$

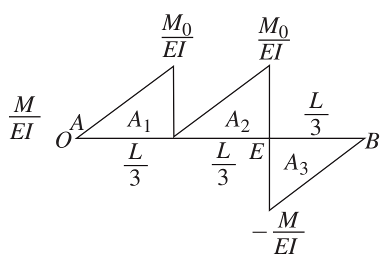
Problem 9.6-11 A simple beam AB is subjected to couples M_0 and $2M_0$ as shown in the figure. Determine the angles of rotation θ_A and θ_B at the the beam and the deflection δ at point D where the load M_0 is applied.



Solution 9.6-11 **Simple beam with two couples**

DEFLECTION CURVE AND M/EI DIAGRAM





$$A_1 = A_2 = \frac{1}{2}\left(\frac{M_0}{EI}\right)\left(\frac{L}{3}\right) = \frac{M_0L}{6EI} \qquad A_3 = -\frac{M_0L}{6EI}$$

ANGLE OF ROTATION θ_A AT END A

$t_{B/A} = BB_1 =$ first moment of area between A and B with respect to B

$$= A_1\left(\frac{2L}{3} + \frac{L}{9}\right) + A_2\left(\frac{L}{3} + \frac{L}{9}\right) + A_3\left(\frac{2L}{9}\right) = \frac{M_0L^2}{6EI}$$

$$\theta_A = \frac{BB_1}{L} = \frac{M_0L}{6EI} \quad \text{(clockwise)} \longleftarrow$$

ANGLE OF ROTATION θ_B AT END B

$t_{A/B} = AA_1 =$ first moment of area between A and B with respect to A

$$= A_1\left(\frac{2L}{9}\right) + A_2\left(\frac{L}{3} + \frac{2L}{9}\right) + A_3\left(\frac{2L}{3} + \frac{L}{9}\right) = 0$$

$$\theta_B = \frac{AA_1}{L} = 0 \longleftarrow$$

DEFLECTION δ AT POINT D

Distance $DD_1 = \frac{1}{3}(BB_1) = \frac{M_0L^2}{18EI}$

$t_{D_2}/A = D_2D_1 =$ first moment of area between A and D with respect to D

$$= A_1\left(\frac{L}{9}\right) = \frac{M_0L^2}{54EI}$$

$$\delta = DD_1 - D_2D_1 = \frac{M_0L^2}{27EI} \quad \text{(downward)} \longleftarrow$$

NOTE: This deflection is also the maximum deflection.